Classical infinite-range-interaction Heisenberg ferromagnetic model: Metastability and sensitivity to initial conditions

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(Received 24 January 2003; published 19 September 2003)

An N-sized inertial classical Heisenberg ferromagnet, which consists of a modification of the well-known standard model, where the spins are replaced by classical rotators, is studied in the limit of infinite-range interactions. The usual canonical-ensemble mean-field solution of the inertial classical n-vector ferromagnet (for which $n=3$ recovers the particular Heisenberg model considered herein) is briefly reviewed, showing the well-known second-order phase transition. This Heisenberg model is studied numerically within the microcanonical ensemble through molecular dynamics. In what concerns the caloric curve, it is shown that, far from criticality, the kinetic temperature obtained at the long-time-limit microcanonical-ensemble simulation recovers well the equilibrium canonical-ensemble estimate, whereas, close to criticality, a discrepancy (presumably due to finite-size effects) is found. The time evolution of the kinetic temperature indicates that a basin of attraction exists for the initial conditions for which the system evolves into a metastable state, whose duration diverges as $N\to\infty$, before attaining the terminal thermal equilibrium. Such a metastable state is observed for a whole range of energies, which starts right below criticality and extends up to very high energies (in fact, the gap between the kinetic temperatures associated with the metastable and the terminal-equilibrium states is expected to disappear only as one approaches infinite energy). To the best our knowledge, this has never before been observed on similar Hamiltonian models, in a noticeable way, for such a large range of energies. For example, for the XY $(n=2)$ version of the present model, such a behavior was observed only near criticality. It is shown also that the (metastable state) maximum Lyapunov exponent decreases with N like $\lambda_{max}\sim N^{-\kappa}$, where for the initial conditions employed herein (maximal magnetization), $\kappa=0.225\pm0.030$, both above and below the critical point.

DOI: 10.1103/PhysRevE.68.036115 PACS number(s): 05.70.Fh, 05.20.−y, 05.50.+q, 64.60.Fr

I. INTRODUCTION

The Boltzmann-Gibbs (BG) statistical mechanics represents one of the most successful physical theories, providing a good description of many experimental systems at equilibrium [1–3]. The applicability of such a formalism is justified upon the validity of the ergodic hypothesis, which requires that the whole phase space should be equally visited in the infinite-time limit. Typically, this occurs for large Hamiltonian systems, with dynamical variables connected by short-range interactions, leading to a microscopic dynamics characterized by a quasicontinuum Lyapunov spectrum, whose largest value remains positive in the thermodynamic limit. Such a chaotic dynamical behavior leads to a quick occupation of phase space, ensuring a safe use of the ensemble theory, and so the BG statistics may be connected to the standard extensive thermodynamics in the well-known elegant manner. However, ergodicity and, consequently, the validity of the standard equilibrium ensembles, depend crucially on the nature of the Hamiltonian system considered [4]. In particular, systems characterized by long-range interactions, or long-range microscopic memory, may present a breakdown of ergodicity, leading to a fractal (or even more complex) structure in phase space. In such cases, the BG statistical-mechanics framework looses its validity and some more general theory must be employed. Recently, a large variety of evidences has been presented, exhibiting results that do not conform with the BG formalism; one may mention observations on turbulent plasmas [5], turbulent fluids [6–8], astrophysical systems [9–13], quantum chaos [14], logistic map [15], glasses [16,17], and complex systems [18,19], among others. Such systems exhibit evident inconsistencies with one of the main characteristics in the BG formalism, which is the extensivity of the entropy and the exponential weight factor associated with it. For an N-particle system, the extensivity property means that quantities such as the internal or free energy *per particle* should approach a well-defined finite thermodynamic limit when $N\to\infty$. When long-range interactions are present, each microscopic constituent of the system interacts with all the others, leading to an energy that depends more than linearly on N, and, obviously, to a nonextensive behavior. As a consequence of this, additivity does not hold, in the sense that the free energy of the whole system is not equal to the sum of the free energies of its macroscopic parts, and the application of the BG formalism becomes questionable. A generalized nonextensive thermostatistical formalism, proposed over a decade ago [20,21], seems to be a good candidate to deal with such systems.

Recently, a lot of attention has been dedicated to a classical Hamiltonian system, namely, the inertial long-range-

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interaction XY model, which consists of an assembly of N classical planar rotators interacting through a long-range potential [22–35]. Such a system, which has been investigated numerically within the microcanonical ensemble, presented clear indications that a more general statistical-mechanics formalism is required for its description. In particular, for the case of infinite-range interactions, i.e., mean-field limit, for which a well-known continuous phase transition occurs, if one considers a total energy close to and below the critical energy, there exists a basin of attraction for the initial conditions for which the system gets captured in a metastable state, whose duration increases with N, before attaining the terminal thermal equilibrium. Therefore, if one considers the thermodynamic limit ($N\to\infty$) *before* the long-time limit, the system will remain in the metastable state and will never reach the terminal equilibrium state, in such a way that the phase space will *not* be equally and completely covered. Moreover, in such a metastable state, the maximum Lyapunov exponent approaches zero, as $N\to\infty$, contrary to what is expected in a standard BG equilibrium state.

In the present work we perform molecular dynamical investigations, of the isolated inertial infinite-range-interaction Heisenberg ferromagnet, defined by N classical Heisenberg-like rotators. We show that the metastable state that occurs in the corresponding XY version of the model, near criticality, now appears in a noticeable way for a much wider extent, for an energy range that starts right below criticality and prolongs up to very high energies. In the following section, we review the equilibrium canonical-ensemble solution of the model. In Sec. III we present the results of our numerical investigation, and finally, in Sec. IV, we present our main conclusions.

II. THE EQUILIBRIUM CANONICAL-ENSEMBLE SOLUTION

Let us now work out the equilibrium canonical-ensemble solution of the model. For the sake of generality, throughout this section, we will deal with an inertial n-vector ferromagnet, composed of N classical rotators, each of them defined in an n-dimensional configurational space. The Hamiltonian of the system is given by

$$H=K+V=\frac{1}{2}\sum_{i=1}^{N}\sum_{\mu=1}^{n}L_{i\mu}^2+\frac{1}{2N}\sum_{i,j=1}^{N}(1-\vec{S}_i\cdot\vec{S}_j)$$
$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\mu=1}^{n}L_{i\mu}^2+\frac{1}{2N}\sum_{i,j=1}^{N}\left(1-\sum_{\mu=1}^{n}S_{i\mu}S_{j\mu}\right), \tag{2.1}$$

where the index μ ($\mu=1,2,\ldots,n$) denotes Cartesian components and $L_{i\mu}$ represents the μ component of the angular momentum (or the rotational velocity, since we are assuming unit inertial moments) of rotator i, when $n=3$; when n is different from 3, $L_{i\mu}$ can be considered as a proper generalized momentum. In fact, for the general case of n components (as considered in the present section), $\vec{L}$ is an antisymmetric tensor of rank 2, with $n(n-1)/2$ components. This number coincides with n for $n=3$, allowing us to treat $\vec{L}$ as a (pseudo)vector for $n=3$.

It is important to remind that the N dependence in the coupling constant above, is usually (but not necessarily, see Ref. [24]) introduced in order to yield a sensible thermodynamic limit, i.e, a finite free energy per particle when $N\to\infty$, within the equilibrium ensemble theory of standard statistical mechanics.

The rotators are allowed to vary their directions continuously inside an n-dimensional sphere of unit radius, leading to the constraint

$$\sum_{\mu=1}^{n}S_{i\mu}^2=1\quad(i=1,2,\ldots,N). \tag{2.2}$$

It should be mentioned that such a constraint reduces the number of degrees of freedom per particle to $n-1$, in such a way that the total number of degrees of freedom of the system is given by $N(n-1)$. The model defined above recovers, as particular cases, the mean-field inertial XY (whose equilibrium canonical-ensemble solution was presented in Ref. [22]) and Heisenberg (whose dynamics will be discussed in the following section) models, for $n=2$ and $n=3$, respectively. Although our model is composed of one-dimensional inertial constituents (rotators), we shall sometimes refer to $\vec{S}\equiv(S_1,S_2,\ldots,S_n)$ as spin variables, considering the close analogy of the above-defined model with the standard n vector ferromagnet.

One may now follow the standard procedure by rewriting the Hamiltonian as

$$H=\frac{N}{2}+\frac{1}{2}\sum_{i=1}^{N}\sum_{\mu=1}^{n}L_{i\mu}^2-\frac{1}{2N}\sum_{\mu=1}^{n}\left(\sum_{i=1}^{N}S_{i\mu}\right)^2, \tag{2.3}$$

in such a way that the partition function becomes

$$Z=\exp\left(\frac{-\beta N}{2}\right)\int\left(\prod_{i=1}^{N}\prod_{\mu=1}^{n}dS_{i\mu}\right)\left(\prod_{i=1}^{N}\prod_{\mu=1}^{n-1}dL_{i\mu}\right)$$
$$\times\left[\prod_{i=1}^{N}\delta\left(\sum_{\mu=1}^{n}S_{i\mu}^2-1\right)\right]\left[\prod_{i=1}^{N}\prod_{\mu=1}^{n-1}\exp\left(\frac{-\beta L_{i\mu}^2}{2}\right)\right]$$
$$\times\left\{\prod_{\mu=1}^{n}\exp\left[\left(\frac{\beta}{2N}\right)\left(\sum_{i=1}^{N}S_{i\mu}\right)^2\right]\right\}. \tag{2.4}$$

The constraints of Eq. (2.2) are taken into account in the equation above, through the δ functions, as well as in the products over the angular momentum variables, which apply only over the effective number of degrees of freedom per particle, $\mu=1,2,\ldots,n-1$. The squared Σ_i may be linearized through the application of a Hubbard-Stratonovich-like transformation [2], which introduces a set of parameters $\{x_\mu\}$; rescaling $\{x_\mu\}\to\sqrt{N}\{x_\mu\}$ one gets

$$Z=\left(\frac{2\pi}{\beta}\right)^{N(n-1)/2}\exp\left(\frac{-\beta N}{2}\right)\int\left\{\prod_{\mu=1}^{n}\left[\left(\frac{N}{2\pi}\right)^{1/2}dx_{\mu}\times\exp\left(\frac{-Nx_{\mu}^{2}}{2}\right)\right]\right\}\prod_{i=1}^{N}\left\{\int\prod_{\mu=1}^{n}\left[dS_{i\mu}\exp(\beta^{1/2}S_{i\mu}x_{\mu})\right]\times\delta\left(\sum_{\mu=1}^{n}S_{i\mu}^{2}-1\right)\right\}. \tag{2.5}$$

As usual, the site index may be discarded, and straightforward calculations lead to

$$Z=\left(\frac{2\pi}{\beta}\right)^{N(n-1)/2}\exp\left(\frac{-\beta N}{2}\right)\int\left[\prod_{\mu=1}^{n}\left(\frac{N}{2\pi}\right)^{1/2}dx_{\mu}\right]\times\exp\left[-N\left(\sum_{\mu=1}^{n}\frac{x_{\mu}^{2}}{2}-\ln\xi\right)\right], \tag{2.6}$$

where

$$\xi=2^{(n-2)/2}\pi^{n}y^{-(n-2)/2}I_{(n-2)/2}(y),\qquad y=\left(\beta\sum_{\mu=1}^{n}x_{\mu}^{2}\right)^{1/2}, \tag{2.7}$$

with $I_k(y)$ denoting modified Bessel functions of the first kind of order k. Considering N large, one may use steepest descents

$$Z\approx\left(\frac{2\pi}{\beta}\right)^{N(n-1)/2}\exp\left(\frac{-\beta N}{2}\right)\times\exp\left[-N\max_{y}\left(\frac{y^{2}}{2\beta}-\ln\xi(y)\right)\right], \tag{2.8}$$

with the condition of maximum leading to the self-consistent equation

$$\overline{y}=\beta\left(\frac{1}{\xi}\frac{d\xi}{dy}\right)_{y=\overline{y}}=\beta\frac{I_{n/2}(\overline{y})}{I_{(n-2)/2}(\overline{y})}. \tag{2.9}$$

In the thermodynamic limit, one obtains the free energy per particle

$$\beta f=\frac{\beta}{2}-\frac{n-1}{2}\ln\left(\frac{2\pi}{\beta}\right)+\frac{\overline{y}^{2}}{2\beta}-\ln\xi(\overline{y}), \tag{2.10}$$

as well as the internal energy per particle

$$u=\frac{n-1}{2\beta}+\frac{1}{2}(1-\vec{m}^{2}), \tag{2.11}$$

where $\vec{m}$ represents the magnetization per particle, whose modulus is directly related to the parameter $\overline{y}$,

$$m\equiv|\vec{m}|=\frac{\overline{y}}{\beta}=\frac{I_{n/2}(\beta m)}{I_{(n-2)/2}(\beta m)}. \tag{2.12}$$

The critical temperature of the model may be obtained by considering m small and expanding the right-hand side of Eq. (2.9) in power series (we work in units of $k_B=1$),

$$T_c=\frac{1}{2}\frac{\Gamma(n/2)}{\Gamma((n+2)/2)}=\frac{1}{n}, \tag{2.13}$$

which may be substituted into Eq. (2.11) to yield the critical energy density

$$u_c=\frac{1}{2}[1+(n-1)T_c]=1-\frac{1}{2n}. \tag{2.14}$$

The above results recover those obtained in Ref. [22], for $n=2$, e.g., the self-consistent equation

$$\overline{y}=\beta\frac{I_1(\overline{y})}{I_0(\overline{y})}, \tag{2.15}$$

leading to $T_c=1/2$ and $u_c=3/4$. For the case $n=3$, one gets the well-known self-consistent equation

$$m=\coth(\beta m)-\frac{1}{\beta m}, \tag{2.16}$$

with $T_c=1/3$ and $u_c=5/6$. For n increasing from unity (Ising model) to infinity (spherical model), T_c decreases from 1 to 0 and u_c increases from 1/2 to 1.

In the following section we discuss the dynamics of the particular case $n=3$ of the model defined above.

III. DYNAMICS OF THE MEAN-FIELD INERTIAL HEISENBERG MODEL

A. Molecular dynamics

In this section we will present the results obtained by simulations of the constant-energy dynamics of the model defined by Eqs. (2.1) and (2.2) for the particular case $n=3$, i.e., Heisenberg-like rotators. From now on, we will use the standard notation for the Cartesian components of a Heisenberg model, i.e., $\mu=x,y,z$. The results to be discussed below were obtained by a direct integration of the equations of motion

$$\dot{\vec{L}}_i=\vec{S}_i\times\left(\frac{1}{N}\sum_{j=1}^{N}\vec{S}_j\right)\quad(i=1,2,\ldots,N), \tag{3.1a}$$

$$\dot{\vec{S}}_i=\vec{L}_i\times\vec{S}_i\quad(i=1,2,\ldots,N), \tag{3.1b}$$

which correspond to a set of $6N$ equations to be solved numerically.

It should be stressed that Eqs. (3.1) are not canonical equations of motion, since $\{L_{i\mu}\}$ and $\{S_{i\mu}\}$ are not canonically conjugate variables. In that sense, one could say that Eq. (2.1) represents the energy, and not the Hamiltonian, from which one obtains the canonical equations of motion. In addition to that, according to the discussion of the preceding section, the generalization of Eqs. (3.1) for $n\neq3$ is to be handled with care.

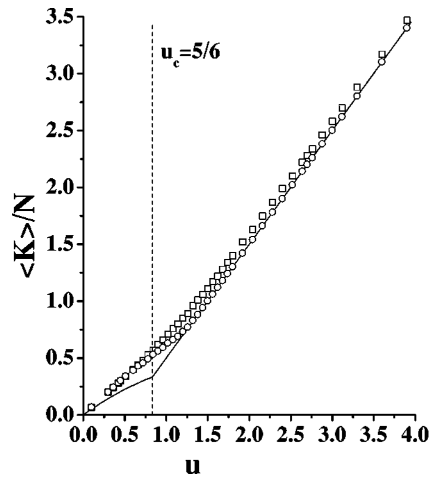

FIG. 1. The caloric curve for the inertial mean-field ferromagnetic Heisenberg model obtained by the equilibrium canonical-ensemble solution (full line). In the vertical axis, $\langle K\rangle/N$ is a quantity that is expected to coincide with the temperature, at equilibrium. The dashed vertical line signals the second-order phase transition critical energy density, $u_c=5/6$. The empty squares and circles represent the estimates of $\langle K\rangle/N$ from the microcanonical-ensemble numerical analysis of a system of size $N=400$, at the metastable state and after that, respectively. For the Hamiltonian defined in Eq. (2.1), energies are dimensionless quantities.

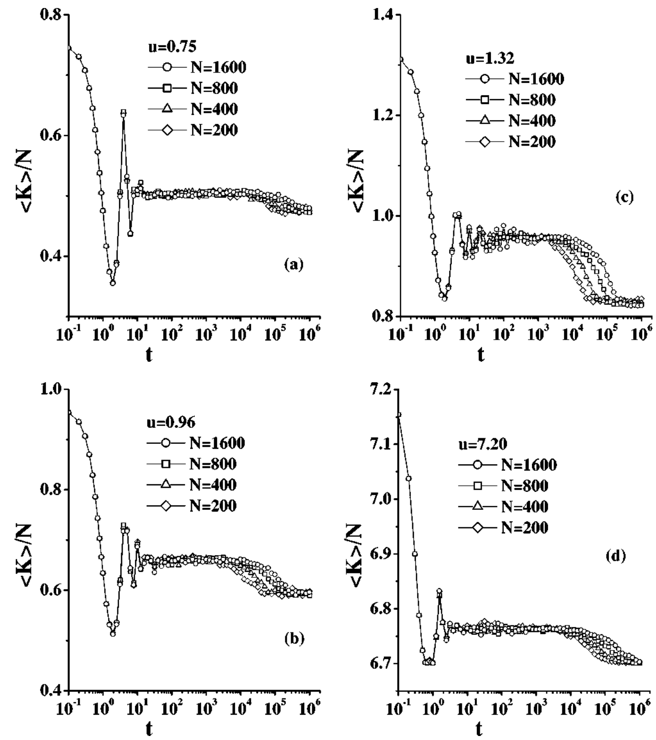

FIG. 2. The microcanonical time evolution of $\langle K\rangle/N$ is represented for several system sizes and different energy densities: just below criticality [$u=0.75$ (a)], just above criticality [$u=0.96$ (b)], at the region where the gap between the metastable and terminal-equilibrium states is maximum [$u=1.32$ (c)] and for $u\gg u_c$ [$u=7.2$ (d)]. The initial conditions are water bag for velocities and $m=1$. For the Hamiltonian defined in Eq. (2.1), energies are dimensionless quantities. The time is also dimensionless and each unit of (physical) time t corresponds to 20 iterations of the equations of motion.

For solving such a set of equations we have used a fourth-order Runge-Kutta-Merson integrator [36] with a time step of 0.05, leading, respectively, to the relative energy and spin-normalization conservations of 10^{-4} and 10^{-3}, or better. The total initial kinetic energy was divided into three equal parts, each of them to be assigned to a given set of Cartesian components of angular velocities $\{L_{i\mu}\}$ $(i=1,2,\ldots,N)$. We have always started the system with the so-called water-bag initial conditions [29,30,34] for each set of components of angular velocities, i.e., each set $\{L_{i\mu}\}$ was extracted from a symmetric uniform distribution and then, translated and rescaled to have zero total momentum. In what concerns the spin variables, we have started our simulations with all spins aligned along the z axis (zero initial potential energy). Our measured quantities correspond to averages over N_s distinct samples, i.e., different initial sets of $\{L_{i\mu}\}$.

B. Caloric curve and metastability

In Fig. 1 we exhibit the caloric curve (full line) obtained by solving the equilibrium canonical-ensemble equations [Eqs. (2.11) and (2.12)]. We have chosen four particular values of the energy density to investigate, within our microcanonical-ensemble molecular dynamical approach, how $\langle K\rangle/N$ evolves in time, for different values of N (it should be mentioned that the quantity $\langle K\rangle/N$, which represents an average over different initial conditions of the kinetic energy per particle, when evaluated at the $t\to\infty$ equilibrium, is expected to coincide with the temperature). Two of the chosen energies, $u=0.75$ [Fig. 2(a)] and $u=0.96$ [Fig. 2(b)], correspond, respectively, to values slightly below and above the critical internal energy $(u_c=5/6)$. The energy $u=1.32$ [Fig. 2(c)] is inside a range of energies where the kinetic temperature $\langle K\rangle/N$ presents a maximum discrepancy between its values at intermediate and long times. Finally, the fourth chosen energy, $u=7.2$ [Fig. 2(d)] corresponds to a value far above u_c. In all the plots of Figs. 2(a)–(d) the system was started with the above-mentioned initial conditions and we have considered $N_s=16$ $(N=200)$, $N_s=12$ $(N=400)$, $N_s=8$ $(N=800)$, and $N_s=4$ $(N=1600)$. We have observed that, after a short transient, the system rapidly reached a metastable or quasistationary state (QSS), with a value of $\langle K\rangle/N$ *higher* than the one predicted by the canonical-ensemble equilibrium theory. It is important to remind that a QSS has been found also for the corresponding XY version of the present model, in an unambiguous way, only near criticality [23,25,27,29,30,34]. Except for very low energies, the QSS is always detected easily in the Heisenberg case; this is shown in Fig. 1, where we exhibit the values of $\langle K\rangle/N$ for both QSSs (empty squares) and terminal-equilibrium states (empty circles), for systems with $N=400$ and different values of the energy density. One clearly sees that, in what concerns the value of $\langle K\rangle/N$ at long times, the thermodynamic limit may be attained within our computational effort (in the sense that the present microcanonical-ensemble numerical approach agrees with the equilibrium

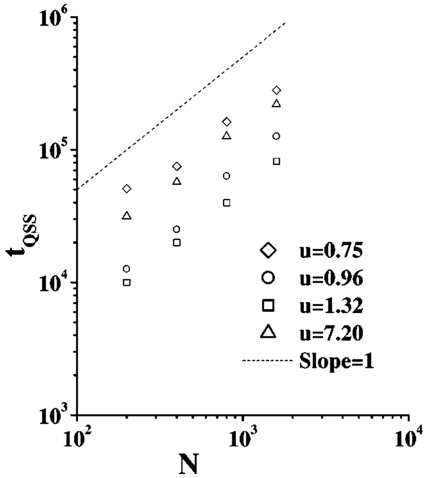

FIG. 3. Log-log plots of the lifetime (t_{QSS}) of the QSS as a function of N, for the energy densities considered in Figs. 2(a)–(d). Simple linear fits yield the slopes 0.99 ± 0.05 ($u=0.75$), 1.10 ± 0.06 ($u=0.96$), 1.01 ± 0.02 ($u=1.32$), and 0.95 ± 0.06 ($u=7.20$). In each case, the slope is very close to 1 (represented by the dashed line), in such a way that $t_{QSS}\sim N$ in all cases. The initial conditions are water bag for velocities and $m=1$.

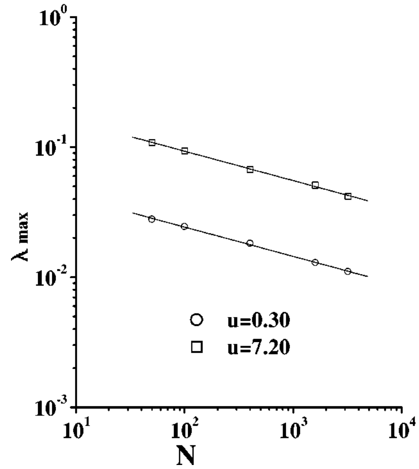

FIG. 4. Log-log plots of the maximum Lyapunov exponent λ_{max} vs N, for energies densities below ($u=0.3$) and above the critical point ($u=7.2$), showing the scaling $\lambda_{max}\sim N^{-\kappa}$. The slopes are essentially the same, within the error bars, yielding $\kappa=0.225\pm0.030$. For $u=7.2$, the distinction between the metastable and terminal equilibrium states is very clear, and this value of κ corresponds to the metastable one. For $u=0.3$, the distinction in not very clear, and this value of κ presumably corresponds to the terminal-equilibrium state.

canonical-ensemble results) for $u\gg u_c$, whereas for $u\sim u_c$ (typically in the range $0.25<u<1.25$), one observes a discrepancy between the numerics and the analytical BG result. In fact, the terminal-equilibrium value of $\langle K\rangle/N$ seems to reach its thermodynamic limit for very small systems, if $u\gg u_c$ [as shown in Fig. 2(d), for the case $u=7.2$, where for the size $N=200$ the thermodynamic limit appears to be attained]. We find two possible causes for explaining the discrepancies shown in Fig. 1, near criticality: (i) strong finite-size effects, in such a way that one should run much larger sizes than those exhibited in Figs. 2(a) and 2(b) in order to reach the thermodynamic-limit values and (ii) a three-plateaux structure, i.e., the long-time states exhibited in Figs. 2(a) and 2(b) are still metastable and the terminal-equilibrium state should occur for much larger times. The clarification of this point naturally deserves further effort, which is out of the scope of the present work.

Although large fluctuations (as time evolves) may be observed in the values of $\langle K\rangle/N$ in the QSS, it appears evident that the gap with respect to the corresponding terminal-equilibrium-state values survives in the thermodynamic limit. If one defines the lifetime of the QSS t_{QSS} as the time at which $\langle K\rangle/N$ presents its halfway between the values at the QSS and the terminal thermal equilibrium, one concludes that such a quantity increases, essentially, linearly with N, as shown in Fig. 3, for the energies considered in Figs. 2(a)–(d). Therefore, the duration of the QSS increases with N, in such a way that, if the thermodynamic limit is performed *before* the long-time limit, the system will never relax to the terminal thermal equilibrium.

C. Sensitivity to the initial conditions

Let us now investigate how the maximal Lyapunov exponent λ_{max} scales with N. Herein we shall use the well-known method for calculating such a quantity by considering the limit [37]

$$\lambda_{max}=\lim_{t\to\infty}\frac{1}{t}\ln\left[\frac{d(t)}{d(0)}\right]=\lim_{t\to\infty}\lambda(t), \tag{3.2a}$$

$$d(t)=\left\{\sum_{i=1}^{N}\sum_{\mu=x,y,z}[(\delta L_{i\mu})^2+(\delta S_{i\mu})^2]\right\}^{1/2}, \tag{3.2b}$$

where $d(t)$ represents the metric distance calculated from infinitesimal displacements in phase space, at time t. We have carried simulations up to $t=10\,000$ (i.e., 200 000 time steps), for different system sizes ($N=50$, 100, 400, 1600, 3200), and λ_{max} was obtained after averaging over $N_s=50$ samples for the smallest size ($N=50$), whereas for all other sizes we have considered $N_s=10$. For the cases where there is an apparent QSS, the time interval considered ensures that the λ_{max} computed does indeed correspond to a quantity in the QSS [see, e.g., Figs. 2(a)–(d)], whereas for the cases where there is no evident QSS, the time used is expected to be sufficient for the system to have reached its terminal thermal equilibrium. As shown in Fig. 4, one has that $\lambda_{max}\sim N^{-\kappa}$, similarly to what happens for the XY version of the present model [23,30,34,35]. We computed κ for energies below u_c ($u=0.30$), as well as for $u\gg u_c$ ($u=7.2$), with the above-mentioned initial conditions; we have obtained essentially the same estimate in both cases, $\kappa=0.225\pm0.030$. We have also computed the exponent κ near criticality and found values that are not in the same universality class of those far from criticality. In fact, preliminary estimates for $u=0.96$ and $u=1.32$ suggest $\kappa=0.15\pm0.02$. Whether such estimates are spurious, due to the nearness of the critical point, or are in some way related to a possible three-plateaux structure, turns out to be a point that deserves further investigation. It is important to remind that, in the mean-field inertial XY model, one has $\kappa=1/3$ for $u \gg u_c$ [23,30,34], whereas $\kappa=1/9$ for $u<u_c$ [35]; in the former case, there is no apparent QSS, in such a way that the estimate $\kappa=1/3$ is expected to apply to a terminal thermal equilibrium state, whereas in the latter, κ was obtained at the peculiar QSS. Such estimates find no similarity with the ones far from criticality presented herein for the Heisenberg case, which apply to the QSS for $u \gg u_c$, whereas for the low energy considered ($u=0.30$) in the case $u<u_c$ we have found no clear evidence of a QSS (although the presence of a QSS, with a small gap, indiscernible due to fluctuations in $\langle K \rangle /N$, with respect to the terminal-equilibrium state, is not ruled out). However, the common feature observed for both XY and Heisenberg models, $\lambda_{\max} \sim N^{-\kappa}$ (yielding a zero maximal Lyapunov exponent in the thermodynamic limit), which seems to hold in the presence or not of an evident QSS, does certainly contradict the standard BG theory for an equilibrium state (which requires a finite $\lambda_{\max}$ in the thermodynamic limit).

IV. CONCLUSION

We have analyzed a system of N Heisenberg-like classical rotators with ferromagnetic infinite-range interactions. The dynamics of the model was studied within the microcanonical ensemble by directly solving the equations of motion. For a finite N, the time evolution of the kinetic temperature shows that there is a basin of attraction for the initial conditions for which the system gets caught in a metastable state, before reaching the terminal-thermal equilibrium. We have shown that the duration of such a metastable state diverges as $N \to \infty$, typically linearly with N. Therefore, if the thermodynamic limit is considered before the long-time limit, the system will never relax to the terminal thermal equilibrium. We have also calculated the maximum Lyapunov exponent above and below the critical point; in both cases, the scaling, $\lambda_{\max} \sim N^{-\kappa}$, was verified. For the particular initial conditions considered (maximal magnetization), the exponent κ presents the same value for energies chosen above and below the critical point, $\kappa=0.225 \pm 0.030$. Above the critical point our estimate applies to the metastable state, whereas the estimate below the critical point is expected to hold for the terminal thermal equilibrium, since in such a case we have found no clear evidence of the existence of a metastable state. Preliminary studies suggest that, above criticality, the exponent κ does not depend, within the error bars, on the initial conditions employed; however, below criticality, different initial conditions may possibly lead to a breakdown of universality, with different estimates for κ. In particular, the κ estimates below criticality seem to vary if the initial conditions for the spin variables break or not the Heisenberg-like symmetry of the system, i.e., if we start with $m \neq 0$ (present paper) or $m=0$. Further studies to clarify this point constitute the next step along the present lines.

ACKNOWLEDGMENTS

We thank C. Anteneodo, E. P. Borges, and E. M. F. Curado for fruitful conversations. The partial financial supports from CNPq, Pronex/MCT, and FAPERJ (Brazilian agencies) are acknowledged. One of us (F.D.N.) acknowledges CBPF (Centro Brasileiro de Pesquisas Físicas) for the warm hospitality during a visiting period in which this work was accomplished.

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